cf Koch

§ topological quantum field theory (TQFT)

Atiyah, Segal around '90

Def. d-dimensional TQFT is the following assignment:

- Σ: $(d-1)$ dim'l oriented compact mfd without bdry

  $\longmapsto Z(\Sigma)$: $\mathbb{C}$-vector space

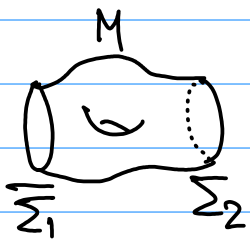
  s.t. • $Z(\Sigma \sqcup \Sigma') = Z(\Sigma) \otimes Z(\Sigma')$

  • $Z(\phi) = \mathbb{C}$

- "cobordism"

  M : d-dim'l oriented compact mfd

  with "in" bdry $\overline{\Sigma_1}$, "out" bdry Σ_2



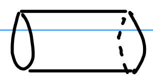
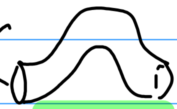
$\longmapsto$ $Z(M) : Z(\Sigma_1) \longrightarrow Z(\Sigma_2)$ linear map

(M: no bdry $\Rightarrow$ $Z(M) : Z(\phi) \longrightarrow Z(\phi)$
$\in \mathbb{C}$ $\quad \| \quad \mathbb{C} \quad$ " $\mathbb{C}$)

s.t. • equivalent cobordism ($M \xrightarrow{\varphi} M'$ diffeo. s.t. $\cup\ \overline{\Sigma_1}, \Sigma_2$ $\cup$ $\varphi|_{\Sigma_i} = id_{\Sigma_i}$)

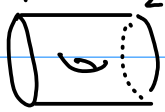
$\Longrightarrow Z(\Sigma_1) \longrightarrow Z(\Sigma_2)$

$Z(M) = Z(M')$

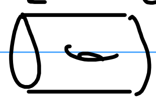
i.e. $Z($  $) = Z($  $)$

• $Z(\Sigma \times [0,1]) = id_{Z(\Sigma)}$

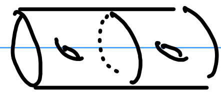
• $\overline{\Sigma_1}$ Σ_2 $\overline{\Sigma_2}$ Σ_3



M_1



M_2



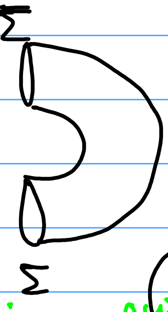
$\underbrace{\qquad}_{M = M_1 \cup_{\Sigma_2} M_2}$

$$\begin{array}{ccc} Z(M) : Z(\Sigma_1) & \longrightarrow & Z(\Sigma_3) \\ Z(M_1) \searrow & \circlearrowleft & \nearrow Z(M_2) \\ & Z(\Sigma_2) & \end{array}$$

tensor

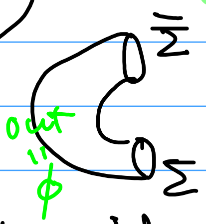
• $Z(M \sqcup M') : Z(\Sigma_1)\otimes Z(\Sigma_1') \longrightarrow Z(\Sigma_2)\otimes Z(\Sigma_2')$
$\qquad \| \quad Z(M)\otimes Z(M')$

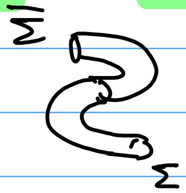
Consider



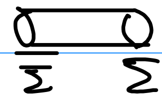
$$Z(\bar\Sigma)\otimes Z(\Sigma) \xrightarrow{<\,,\,>} \mathbb{C} \quad \text{pairing}$$

$$\gamma : \mathbb{C} \longrightarrow Z(\bar\Sigma)\otimes Z(\Sigma)$$





$\underset{\text{equiv}}{\cong}$



$$\therefore \quad id_{Z(\Sigma)} = (<\,,\,>\otimes id_{Z(\Sigma)})\circ(id_{Z(\Sigma)}\otimes\gamma)$$

$$\gamma(1) = \sum u_i \otimes v_i \quad \Rightarrow \quad v = \sum <v, u_i> v_i$$

Prop $\quad Z(\Sigma)$ is finite dim'l and $<\,,\,>$ is non-degenerate pairing $\quad \therefore Z(\bar\Sigma) = Z(\Sigma)^*$

$Z(\Sigma)$ is called the quantum Hilbert space ass. with Σ

Rem. not necessarily positive definite

This axiomatic approach was introduced by Segal, Atiyah so that mathematicians could understand

Witten's topological quantum field theory

based on usual quantum field theory

§ Intuitive idea behind the axiomatic approach

X : target space (nothing to do with Σ, M etc)
(σ - model)

Suppose $L : \mathrm{Map}(M, X) \longrightarrow \mathbb{R}$ action

lagrangian φ e.g. $M = S^1 = \mathbb{R}/\mathbb{Z}$ φ

$$L(\varphi) = \int_0^1 \left|\frac{d}{dt}\varphi\right| dt$$

$$\rightsquigarrow Z(M) \text{ " $=$ " } \int_{\mathrm{Map}(M,X)} e^{-L(\varphi)} D\varphi \in \mathbb{C}$$

Feynmann path integral

Next suppose M has "out" bdry Σ $f : \Sigma \longrightarrow X$

$$Z(M, f) \text{ " $=$ " } \int_{\substack{\mathrm{Map}(M,X) \ni \varphi \\ \varphi|_\Sigma = f}} e^{-L(\varphi)} D\varphi$$

We regard this as a function

$$\begin{array}{ccc} f & \longmapsto & Z(M, f) \\ \cap & & \uparrow \\ \mathrm{Map}(\Sigma, X) & \longrightarrow & \mathbb{C} \end{array}$$

i.e. $Z(M, \cdot) \in \mathrm{Function}(\mathrm{Map}(\Sigma, X))$

$\uparrow$
$\therefore$ $\mathbb{C}$ - vector space

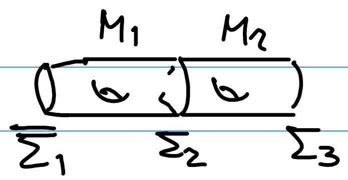
$\mathrm{Map}(\Sigma, X)$, $\Sigma = \Sigma_a \cup \Sigma_b$ $\Rightarrow$ $\mathrm{Map}(\Sigma_a, X) \times \mathrm{Map}(\Sigma_b, X)$

Recall $Z(\Sigma)$ quantum Hilbert space
$\parallel$

M $\rightsquigarrow Z(M, \cdot) \in \mathrm{Func}(\mathrm{Map}(\Sigma, X))$
with $\partial M = \Sigma$

$Z(M) \in \mathrm{Hom}(\underset{\mathbb{C}}{\underset{\parallel}{Z(\phi)}}, Z(\Sigma)) = Z(\Sigma)$

cobordism



$\longrightarrow X$

$f_1 : \Sigma_1 \longrightarrow X \qquad\qquad f_3 : \Sigma_3 \longrightarrow X$

$$Z(M_1 \cup_{\Sigma_2} M_2, f_1, f_3) \text{``}=\text{''} \int_{\substack{\mathrm{Map}(M,X) \ni \varphi \\ \varphi|_{\Sigma_1} = f_1,\ \varphi|_{\Sigma_3} = f_3}} e^{-L(\varphi)} D\varphi$$

Note :

$$\mathrm{Map}(M_1 \cup_{\Sigma_2} M_2, X) = \mathrm{Map}(M_1, X) \underset{\mathrm{Map}(\Sigma_2, X)}{\times} \mathrm{Map}(M_2, X)$$

$$= \{ (\varphi_{12}, \varphi_{23}) \in \mathrm{Map}(M_1, X) \times \mathrm{Map}(M_2, X) \mid \varphi_{12}|_{\Sigma_2} = \varphi_{23}|_{\Sigma_2} \}$$

$\therefore\ Z(M_1 \cup_{\Sigma_2} M_2, f_1, f_3)$

$$= \int Df_2 \underbrace{\int_{\substack{\mathrm{Map}(M_1,X) \ni \varphi_{12} \\ \varphi_{12}|_{\Sigma_1} = f_1 \quad \varphi_{12}|_{\Sigma_2} = f_2}} e^{-L(\varphi_{12})} D\varphi_{12}}_{Z(M_1, f_1, f_2)} \ \underbrace{\int_{\substack{\mathrm{Map}(M_2,X) \ni \varphi_{23} \\ \varphi_{23}|_{\Sigma_3} = f_3 \\ \varphi_{23}|_{\Sigma_2} = f_2}} e^{-L(\varphi_{23})} D\varphi_{23}}_{Z(M_2, f_2, f_3)}$$

$$= \int_{\mathrm{Map}(\Sigma_2, X)} Df_2\, Z(M_1, f_1, f_2) \cdot Z(M_2, f_2, f_3)$$

$Z(\Sigma_1) \longrightarrow Z(\Sigma_2)$

(cf. convolution product of functions

Fourier transform $\hat{f}(\xi) = \int e^{ix\xi} f(x)\, dx$

1-dim'l QFT = Quantum mechanics

$M^1 \longrightarrow X$ .... motion of particles

Σ^0 : 0-dim'l mfd e.g. $\{0\}$ $\quad Map(\{0\}, X) \cong X$

$Func(Map(\{0\},X)) = L^2(X)$

$\longrightarrow$ This case can be path-integral can be rigorously justified

But $Z(\Sigma)$ is a finite dim'l vector space $\neq L^2(X)$

Q. Why is topological QFT meaningful ?

A TQFT is a toy model for supersymmetric QFT's
超対称性

Replace $L^2(X) \rightsquigarrow \Omega^*_{L^2}(X)$: "L^2"-differential form

$\circlearrowleft d$: exterior differential s.t. $d^2 = 0$

$Z(M) : Z(\Sigma_1) \longrightarrow Z(\Sigma_2)$

M // 1 mfd $\quad Z(\Sigma_1) = (\Omega^*_{L^2}(X))^{\otimes \# \Sigma_1} \quad Z(\Sigma_2) = (\Omega^*_{L^2}(X))^{\otimes \# \Sigma_2}$

$\supset \circ \subset$

supersymmetry : $Z(M)$ commutes with d

$$d\, Z(M) = Z(M)\, d$$

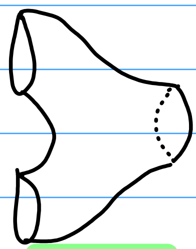
Then $Z(M)$ induces $\mathrm{Ker}\, d / \mathrm{Im}\, d \longrightarrow \mathrm{Ker}\, d / \mathrm{Im}\, d$

$H^*(X)^{\otimes \cdots} \qquad H^*(X)^{\otimes \cdots}$

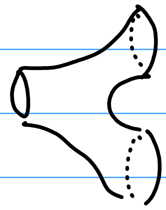
finite dim'l

§ $d=2$ TQFT $Z(S^1) =: V$: vector space

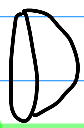
note $\overline{S^1} = S^1$

 : $V\otimes V \xrightarrow{\mu} V$

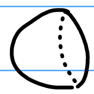
product

 : $V \xrightarrow{\Delta} V\otimes V$

coproduct

 : $V \xrightarrow{\varepsilon} \mathbb{C}$

counit

 : $\mathbb{C} \xrightarrow{1} V$

unit

<u>Folklore theorem</u> $d=2$ TQFT $\longleftrightarrow$ commutative Frobenius algebra

cf. Kock

<u>Def.</u> $(V, 1, \mu, \varepsilon)$: comm. Frobenius algebra

- V : $\mathbb{C}$-vectr sp   finite dim'l
- μ : commutative multiplication on V
  s.t. 1 is unit
- $\varepsilon : V \longrightarrow \mathbb{C}$ Frobenius form
  s.t. $(\ ,\ ) = \varepsilon\circ\mu : V\otimes V \longrightarrow \mathbb{C}$
  is non-degenerate

Δ is the dual of μ via $(\ ,\ )$

<u>ex</u> of Frobenius alg.
comm.

$H^{even}(X)$ (even dim)

$\mu = \cup$

$\varepsilon = \int_X \cdot \; : H^{even}(X) \to \mathbb{C}$

§ クーロン枝

I am recently studying Coulomb branches of gauge theories.

This is partially motivated by $d=3$ TQFT

$A = Z(S^2)$ is a commutative algebra

but I do not assume A is finite dimensional

$$\left(Z(S^2 \sqcup S^2) = Z(S^2) \hat{\otimes} Z(S^2) \right)$$

completed tensor product

Coulomb branch $\mathcal{M}$ is an affine algebraic variety ($/\mathbb{C}$)

defined as $\mathcal{M} = \mathrm{Spec}\, A$

or $\mathbb{C}[\mathcal{M}] = A$

ring of algebraic functions

Originally Coulomb branch was studied by physicists without mathematically rigorous definition.